

Wilson loop remainder function for null polygons in the limit of self-crossing

Harald Dorn, Sebastian Wuttke ¹

*Institut für Physik der Humboldt-Universität zu Berlin,
Newtonstraße 15, D-12489 Berlin, Germany*

Abstract

The remainder function of Wilson loops for null polygons becomes divergent if two vertices approach each other. We apply RG techniques to the limiting configuration of a contour with self-intersection. As a result for the two loop remainder we find a quadratic divergence in the logarithm of the distance between the two approaching vertices. The divergence is multiplied by a factor, which is given by a pure number plus the product of two logarithms of cross-ratios characterising the conformal geometry of the self-crossing.

¹dorn@physik.hu-berlin.de, wuttke@physik.hu-berlin.de

1 Introduction

In recent years a lot of effort has been devoted to the investigation of gluon scattering amplitudes and Wilson loops in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This includes the exposure of the BDS structure [1], the relation of MHV scattering amplitudes to Wilson loops and string surfaces at strong coupling [2] and the verification of this relation also for weak coupling, together with the analysis of dual conformal invariance [3]. The Wilson loops for null n -gons for $n = 4, 5$, via the anomalous dual conformal Ward identity, are fixed to the BDS structure. For $n \geq 6$ appears an additional remainder function which depends only on conformal invariants of the corresponding polygon. At strong coupling this remainder function can be related to the solutions of TBA equations for some Y-system [4], but explicit analytical results are available only for polygons in two-dimensional Minkowski space or for some highly symmetric special cases. At weak coupling, in two loop approximation, the remainder function for a generic hexagon has been calculated in [5] and for an octagon with restricted configurations, which can be embedded in two-dimensional Minkowski space, in [6]. The hexagon result has been confirmed independently [7] via the technique of Wilson loop operator product expansion [8], [7].

In view of the complexity of the direct evaluation of the remainder functions, useful information can come also from the study of some limiting cases of the polygon configuration. In this sense the collinear limit of two adjacent edges has been studied in [9] and has played a role also in the analysis of [4, 8]. Another limiting case, we have in mind in this paper, is the limit in which the polygon becomes self-crossing. As for the collinear limit the renormalisation properties then change qualitatively. In a certain sense, this change is even more radically, since we now have to face operator mixing under renormalisation [10–12].

The idea of making use of this limit and the corresponding modified renormalisation group (RG) equation à la [13] has been developed and applied to the hexagon by Georgiou in [14]. This paper predicted a singular behaviour $\propto \log^3(1 - u_2)$ with a pure imaginary prefactor for the two loop hexagon remainder (depending on three cross ratios u_1, u_2, u_3) in the limit $u_1 = u_3, u_2 \rightarrow 1$. It has been argued, that the origin of this term could be related to the discontinuity of a $Li_4(1 - u_2)$ term in the full remainder function. Although there is such a Li_4 in the meanwhile available complete result of [5], due to the subtleties in analytic continuation and the multi-valued nature [7] of the remainder function, it seems to us that still some effort is needed to check matching of the coefficients. For a continuation to the Regge region of the corresponding scattering amplitude in the $2 \rightarrow 4$ or $3 \rightarrow 3$ channel see [15].

Due to the null condition for the edges, for a hexagon a self-crossing can be realised only via crossing of two opposite edges at a common point, distinct from the vertices. For such crossing edges there is no characteristic free adjustable conformal invariant. This explains the appearance of a pure numeric prefactor of the \log^3 divergence in [14].

We expect a more interesting situation for a self-crossing of the null polygon at a point where two vertices coincide. Then the crossing geometry exhibits free adjustable conformal invariants. Such a situation is possible for octagons and higher polygons. Since up to now the octagon remainder for generic configurations is not available,

our final result will be a substantial prediction for a certain limiting behaviour of this unknown function. In this context one should note that the self-crossing limit cannot be reached within the special octagon configurations for which an analytical result has been obtained in [6]. Furthermore, the anticipated effect of conformal invariants on the prefactor of the self-crossing related divergence indicates, that the origin of this term in the wanted exact remainder function should be visible already without handling the subtleties of analytic continuation.

The logic of our paper will follow the lines of [14]. The bare (dimensionally regularised) Wilson loop is given by the BDS structure plus a remainder function \mathcal{R} . In a generic non-intersecting configuration the remainder in the limit $\epsilon \rightarrow 0$ remains finite, becomes independent of the RG scale μ and depends on conformal invariants of the polygon only. Then it constitutes a part of the renormalised Wilson loop for non-intersecting configurations. Since new divergences appear in a configuration with self-crossing, we expect corresponding short distance singularities in the limit of configurations with self-crossing, both in the well-known contributions from the BDS structure and in the unknown remainder function. Our goal is to find the singularity for the remainder function \mathcal{R} .

To proceed in this direction we study the RG equation for the Wilson loop in a self-crossing configuration. Then the remainder function has poles in ϵ . What remains after subtraction of these poles as contribution to the renormalised self-crossing Wilson loop we call \mathcal{R}_{ren} .² Inserting the known BDS structure one ends with an equation for \mathcal{R}_{ren} , which fixes the dependence on powers of $\log \mu$. Since in dimensional regularisation μ originates exclusively as a factor $\mu^{2\epsilon}$ in combination with the coupling constant g^2 , one then can conclude backward, which poles in ϵ the remainder \mathcal{R} has in the self-crossing situation. The final step will be based on the usual observation that the leading singularities in dimensional regularisation and point splitting regularisation coincide, if $\frac{1}{\epsilon}$ is identified with the logarithm of the distance.

2 RG equation for Wilson loops with self-crossing and cusps

We are interested in Wilson loops for null polygons with $n \geq 8$ vertices x_1, \dots, x_n . For this purpose we first start with polygons \mathcal{C} , which are not of null type, i.e. $p_j^2 \neq 0$, $p_j = x_{j+1} - x_j$, and discuss the light-like limit afterwards. The Mandelstam variables are defined as $s_{jk} = (x_j - x_k)^2$. Let two vertices $x_{\hat{k}}$ and $x_{\hat{l}}$ coincide, with more than two vertices between $x_{\hat{k}}$ and $x_{\hat{l}}$ on both parts of the polygonal contour $\mathcal{C} = \mathcal{C}_{\hat{k}\hat{l}} \cdot \mathcal{C}_{\hat{l}\hat{k}}$, see fig.1. Then, with $\mathcal{U}(\mathcal{C}) = \frac{1}{N} \text{tr} P \exp \left(ig \int_{\mathcal{C}} A_{\mu} dx^{\mu} \right)$ in $SU(N)$ gauge theory,

$$\mathcal{W}_1 = \langle \mathcal{U}(\mathcal{C}) \rangle \quad \text{and} \quad \mathcal{W}_2 = \langle \mathcal{U}(\mathcal{C}_{\hat{k}\hat{l}}) \mathcal{U}(\mathcal{C}_{\hat{l}\hat{k}}) \rangle \quad (1)$$

²The index “ren” will be used to mark the renormalised quantities in the self-crossing situation only.

mix under renormalisation [16], [10–12]

$$\mathcal{W}_b = Z_{bc} Z \mathcal{W}_c^{\text{ren}} . \quad (2)$$

Here Z is the product of the Z -factors for the cusps at the vertices x_l , $l \neq \hat{k}, \hat{l}$ and the matrix Z_{bc} takes care of the UV divergences at the crossing point $x_{\hat{k}} = x_{\hat{l}}$. From (2) one gets in standard manner the RG equation (the β function is zero for $\mathcal{N} = 4$ SYM)

$$\mu \frac{\partial}{\partial \mu} \mathcal{W}_a^{\text{ren}} = -\Gamma_{ab}(g^2, \{\vartheta\}_{\text{cross}}) \mathcal{W}_b^{\text{ren}} - \sum_{k \neq \hat{k}, \hat{l}} \Gamma_{\text{cusp}}(g^2, \vartheta_{k,k-1}) \mathcal{W}_a^{\text{ren}} . \quad (3)$$

The angles $\vartheta_{k,l}$ are defined by ³ $\cosh \vartheta_{k,l} = \frac{p_k p_l - i0}{\sqrt{(p_k^2 - i0)(p_l^2 - i0)}}$ and $\{\vartheta\}_{\text{cross}}$ stands for the six angles at the point of self-intersection.

The anomalous dimension matrix Γ_{bc} is related to the matrix Z_{bc} via $\Gamma_{bc} = \mu \frac{\partial}{\partial \mu} (\log Z)_{bc}$ and has been calculated in [12] up to second order in QCD for a smooth intersection. We are interested in the case where we have two cusps at the intersection point. Direct one-loop calculation leads to ⁴

$$\begin{aligned} \Gamma_{11} &= \frac{g^2}{8\pi^2} \left(\frac{N^2 - 1}{N} (f_{\hat{l}-1, \hat{l}} + f_{\hat{k}-1, \hat{k}} - 2) - \frac{1}{N} (B_1 + i\pi h_{\hat{k}, \hat{l}}) \right) \\ \Gamma_{22} &= \frac{g^2}{8\pi^2} \left(\frac{N^2 - 1}{N} (f_{\hat{k}, \hat{l}-1} + f_{\hat{k}-1, \hat{l}} - 2) - \frac{1}{N} (B_2 + i\pi h_{\hat{k}, \hat{l}}) \right) \\ \Gamma_{12} &= \frac{g^2}{8\pi^2} N (B_1 + i\pi h_{\hat{k}, \hat{l}}) \\ \Gamma_{21} &= \frac{g^2}{8\pi^2} \frac{1}{N} (B_2 + i\pi h_{\hat{k}, \hat{l}}) , \end{aligned} \quad (4)$$

with the abbreviations

$$\begin{aligned} f_{k,l} &:= \vartheta_{kl} \coth \vartheta_{kl} , & h_{\hat{k}, \hat{l}} &:= \coth \vartheta_{\hat{k}-1, \hat{l}-1} + \coth \vartheta_{\hat{k}, \hat{l}} \\ B_1 &:= f_{\hat{k}, \hat{l}-1} + f_{\hat{k}-1, \hat{l}} - f_{\hat{k}, \hat{l}} - f_{\hat{k}-1, \hat{l}-1} \\ B_2 &:= f_{\hat{k}-1, \hat{k}} + f_{\hat{l}-1, \hat{l}} - f_{\hat{k}, \hat{l}} - f_{\hat{k}-1, \hat{l}-1} . \end{aligned} \quad (5)$$

We now turn to the light-like limit $p_k^2 \rightarrow 0, \forall k$. Then all angles $\vartheta_{k,l}$ diverge like

$$\vartheta_{k,l} = \log \frac{2p_k p_l - i0}{\sqrt{(p_k^2 - i0)(p_l^2 - i0)}} , \quad (6)$$

³We use the $i0$ -prescription as induced from that of the standard gluon propagator in position space. It has been argued, that for Wilson loops in correspondence to scattering amplitudes the sign has to be reversed [14, 20].

⁴For a smooth intersection, i.e. $p_{\hat{k}} = \lambda p_{\hat{k}-1}$, $p_{\hat{l}} = \kappa p_{\hat{l}-1}$, $\lambda, \kappa > 0$ one gets back the matrix found in [12] (after adapting the normalisation of \mathcal{W}_2 according to (1), see also comments on this in [14]).

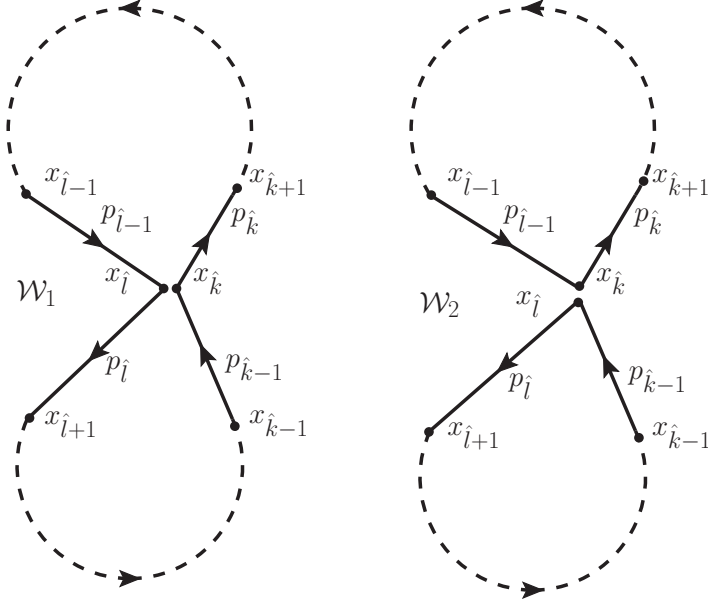


Figure 1: The crossing situation for $x_{\hat{k}} = x_{\hat{l}}$. Both Wilson loops mix under renormalisation. The dashed lines correspond to some further arbitrary light-like polygonal segments of the Wilson loop.

and their hyperbolic cotangent can be replaced by 1. This implies that B_1 and B_2 become logarithms of cross ratios

$$B_1 + 2\pi i = \log \frac{s_{\hat{k}+1, \hat{l}-1} s_{\hat{k}-1, \hat{l}+1}}{s_{\hat{k}+1, \hat{l}+1} s_{\hat{k}-1, \hat{l}-1}}, \quad B_2 + 2\pi i = \log \frac{s_{\hat{k}-1, \hat{k}+1} s_{\hat{l}-1, \hat{l}+1}}{s_{\hat{k}+1, \hat{l}+1} s_{\hat{k}-1, \hat{l}-1}}. \quad (7)$$

In general there are two independent cross ratios associated with four points. In four dimensions the position of $x_{\hat{k}} = x_{\hat{l}}$ is fixed by the light-likeness condition for the four neighbouring edges and the neighbouring points $\{x_{\hat{k}+1}, x_{\hat{l}-1}, x_{\hat{l}+1}, x_{\hat{k}-1}\}$ are not restricted. Thus there are two independent cross ratios describing the crossing situation that we encounter in the matrix Γ_{bc} .

Finally, performing the 't Hooft limit $N \rightarrow \infty$, with $a := \frac{g^2 N}{8\pi^2}$ kept fixed, we arrive at

$$\Gamma_{bc} = a \begin{pmatrix} \vartheta_{\hat{l}-1, \hat{l}} + \vartheta_{\hat{k}-1, \hat{k}} - 2 & B_1 + 2\pi i \\ 0 & \vartheta_{\hat{k}, \hat{l}-1} + \vartheta_{\hat{k}-1, \hat{l}} - 2 \end{pmatrix} + \mathcal{O}(a^2). \quad (8)$$

Note that due to the colour structure Z_{21} and Γ_{21} are zero in all orders of perturbation theory.

In the light-like limit Γ_{11} , Γ_{22} and $\Gamma_{\text{cusp}}(g^2, \vartheta_{k, k-1})$ become divergent and make the RG equation (3) ill defined. According to [13, 17] the anomalous dimension $\Gamma_{\text{cusp}}(g^2, \vartheta)$ for large ϑ has an all order asymptotic behaviour $\Gamma_{\text{cusp}}(g^2, \vartheta) = \vartheta \Gamma_{\text{cusp}}(a) + \mathcal{O}(1)$. Based on this observation in [13], by suitable differentiation with respect to Mandelstam variables and backward integration, a modified RG equation has been derived for Wilson loops for non-intersecting null polygons. The resulting equation can be described by the following recipe: keep the structure of the RG equation and replace every vanishing p_k^2 by $-\frac{1}{\mu^2}$, where μ is the RG scale. In the process of backward

integration a new integration constant appears. It depends on g^2 only. The equation has been checked explicitly on two loop level [18]. Following [14] we assume the same recipe to work also in the case of Wilson loops for self-crossing null polygons. An analogous structure has been obtained in the study of infrared divergences of scattering amplitudes [19].

Our basic RG equation for $\mathcal{W}_1^{\text{ren}}$, obtained with the just described procedure from (3),(8),(6) is then

$$\mu \frac{\partial}{\partial \mu} \log \mathcal{W}_1^{\text{ren}} = -\Gamma_{12} \frac{\mathcal{W}_2^{\text{ren}}}{\mathcal{W}_1^{\text{ren}}} - \left(\Gamma_{11} + \bar{\Gamma}(a) + \frac{\Gamma_{\text{cusp}}(a)}{2} \sum_{k \neq \hat{k}, \hat{l}} \log(-\mu^2 s_{k-1, k+1} + i0) \right). \quad (9)$$

Γ_{cusp} and the new object $\bar{\Gamma}$, arising in the approach of [13] as integration constant, depend on the coupling a only. $\Gamma_{12} = a(B_1 + 2\pi i) + \mathcal{O}(a^2)$. For convenience we understand the p_k independent part of Γ_{11} to be included in $\bar{\Gamma}$ and will use

$$\Gamma_{11} = a \left(\log(-\mu^2 s_{\hat{l}-1, \hat{l}+1} + i0) + \log(-\mu^2 s_{\hat{k}-1, \hat{k}+1} + i0) \right) + \mathcal{O}(a^2). \quad (10)$$

The crucial property of (9) is, that since Γ_{12} starts at order a , to balance the order a^2 of $\log \mathcal{W}_1^{\text{ren}}$, only one loop information on $\mathcal{W}_1^{\text{ren}}$, $\mathcal{W}_2^{\text{ren}}$ is needed on the right hand side.

3 BDS structure and RG equation for the remainder

Taking into account the recursive BDS structure [1, 21], corrected by the remainder function \mathcal{R}_n , the generic n -sided null polygon Wilson loop is given by

$$\log \mathcal{W} = \sum_{l=1}^{\infty} a^l (f^{(l)}(\epsilon) w(l\epsilon) + C_n^{(l)}) + \mathcal{R}_n + \mathcal{O}(\epsilon). \quad (11)$$

Here the $C^{(l)}$ are numbers, $f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$, and $w_n(\epsilon)$ is the one loop contribution

$$w_n(\epsilon) = -\frac{1}{2} \sum_{k=1}^n \frac{1}{\epsilon^2} (-\mu^2 s_{k-1, k+1} + i0)^\epsilon + F_n(\mu^2, \epsilon, s). \quad (12)$$

For a generic null polygon configuration F_n and $\mathcal{R}_n(\mu^2, \epsilon, s) = a^2 \mathcal{R}_n^{(2)}(\mu^2, \epsilon, s) + \dots$ stay finite and become independent of μ^2 in the limit $\epsilon \rightarrow 0$.

Relating $f^{(l)}$ via

$$f_0^{(l)} = \frac{\Gamma_{\text{cusp}}^{(l)}}{2}, \quad f_1^{(l)} = \frac{l}{2} \Gamma^{(l)} \quad (13)$$

to the cusp anomalous dimension and the collinear anomalous dimension, as well as taking into account $f^{(1)}(\epsilon) = 1$ (i.e. $\Gamma_{\text{cusp}}^{(1)} = 2$), $C^{(1)} = 0$ and $\Gamma^{(1)} = 0$ we get up to

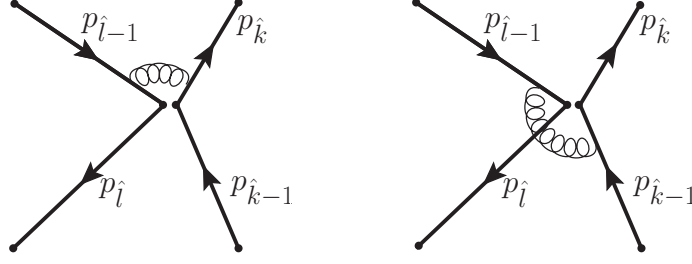


Figure 2: These are two of four diagrams that are responsible for the new divergences in F_n .

two loops [14]

$$\begin{aligned} \log \mathcal{W} = & -\frac{1}{4} \sum_{l=1,2} a^l \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right) \sum_k (-\mu^2 s_{k-1,k+1})^{l\epsilon} + a F_n(\mu^2, \epsilon, s) - \frac{a^2 n}{8} f_2^{(2)} \\ & + a^2 \left(\frac{\Gamma_{\text{cusp}}^{(2)}}{2} F_n(\mu^2, 2\epsilon, s) + \epsilon \Gamma^{(2)} F_n(\mu^2, 2\epsilon, s) + C^{(2)} + \mathcal{R}_n^{(2)}(\mu^2, \epsilon, s) \right) + \mathcal{O}(\epsilon). \end{aligned} \quad (14)$$

The term $\epsilon \Gamma^{(2)} F_n$ has been kept, since in the crossing configuration under discussion F_n develops a pole in ϵ . As a consequence, now the $\mathcal{O}(\epsilon, a^3)$ estimate holds not only in the generic, but also in the limit of a configuration with crossing.

There are three sources for pole terms. The poles of the first term on the r.h.s. are present already in a generic configuration. After expanding the terms $(-\mu^2 s_{k-1,k+1})^{l\epsilon}$ one gets \log^2 and \log terms in momenta as contributions to $\log \mathcal{W}^{\text{ren}}$. The remainder function becomes divergent in the crossing configuration, let us call $\mathcal{R}_n^{(2)\text{ren}}(\mu^2, s)$ what remains after subtraction of the poles in ϵ . The last source for pole terms is the one loop function F_n . The poles of the one loop function F_n in the crossing configuration arise from the diagrams in fig. 2. Note that these diagrams are finite for a generic configuration and that the generic poles of the one loop Wilson loop are taken into account by the first term of the r.h.s. of (14) already. We find

$$\begin{aligned} F_n(\mu^2, \epsilon, s) = & \frac{1}{2\epsilon} \log \frac{s_{\hat{k}-1, \hat{l}-1} s_{\hat{k}+1, \hat{l}+1}}{s_{\hat{k}+1, \hat{l}-1} s_{\hat{k}-1, \hat{l}+1}} \\ & + \frac{1}{4} (L_{\hat{k}-1, \hat{l}-1}^2 + L_{\hat{k}+1, \hat{l}+1}^2 - L_{\hat{k}+1, \hat{l}-1}^2 - L_{\hat{k}-1, \hat{l}+1}^2) + \tilde{F}_n(s) + \mathcal{O}(\epsilon), \end{aligned} \quad (15)$$

where $\tilde{F}_n(s)$ is now independent of μ^2 . In order to improve the readability of our formulæ, we introduced the following abbreviation⁵

$$L_{jk} := \log(-\mu^2 s_{jk} + i0). \quad (16)$$

We now extract from (14) and (15) all the ingredients for the RG equation (9) and start with the quotient $\frac{\mathcal{W}_2^{\text{ren}}}{\mathcal{W}_1^{\text{ren}}}$, which will be needed in one loop approximation only.

⁵For notational convenience we drop the $i0$ terms later on. s_{kl} stands for $s_{kl} - i0$.

In this order $\log \mathcal{W}_1^{\text{ren}}$ is given by minimal subtraction of corresponding poles in ϵ in (14) taking into account (15)

$$\begin{aligned} \log \mathcal{W}_1^{\text{ren}} = & -\frac{a}{4} \left(L_{\hat{k}+1, \hat{l}-1}^2 + L_{\hat{k}-1, \hat{l}+1}^2 - L_{\hat{k}-1, \hat{l}-1}^2 - L_{\hat{k}+1, \hat{l}+1}^2 + \sum_{k=1}^n L_{k-1, k+1}^2 \right) \\ & + a \tilde{F}_n(s) + \mathcal{O}(a^2) . \end{aligned} \quad (17)$$

In the planar limit under discussion, \mathcal{W}_2 for the self-crossing n -gon factorises in the product of two Wilson loops for the two parts, the n_+ -gon $\mathcal{C}_{\hat{k}\hat{l}}$ and the n_- -gon $\mathcal{C}_{\hat{l}\hat{k}}$ ($n_+ + n_- = n$). For these two factors (15) is irrelevant and we get from (14)

$$\begin{aligned} \log \mathcal{W}_2^{\text{ren}} = & -\frac{a}{4} (L_{\hat{k}, \hat{k}+2}^2 + \dots + L_{\hat{k}+1, \hat{l}-1}^2) + aF_{n_+} \\ & -\frac{a}{4} (L_{\hat{l}, \hat{l}+2}^2 + \dots + L_{\hat{k}-1, \hat{l}+1}^2) + aF_{n_-} + \mathcal{O}(a^2) . \end{aligned} \quad (18)$$

Together with (17) the last equation implies

$$\begin{aligned} \frac{\mathcal{W}_2^{\text{ren}}}{\mathcal{W}_1^{\text{ren}}} = & 1 + \frac{a}{4} \left(L_{\hat{l}-1, \hat{l}+1}^2 + L_{\hat{k}-1, \hat{k}+1}^2 - L_{\hat{k}-1, \hat{l}-1}^2 - L_{\hat{k}+1, \hat{l}+1}^2 \right) \\ & + a (F_{n_+} + F_{n_-} - \tilde{F}_n) + \mathcal{O}(a^2) . \end{aligned} \quad (19)$$

F_{n_+} , F_{n_-} and \tilde{F}_n are independent of μ^2 for $\epsilon = 0$. For the l.h.s. of (9) we need $\log \mathcal{W}_1^{\text{ren}}$ at order a^2 . If there was no mixing with \mathcal{W}_2 we would get the $\mathcal{O}(a^2)$ contribution, similar to the lowest order, by minimal subtraction of ϵ -poles in (14), (15). To take care of the mixing effect, let us denote ($b = 1, 2$)

$$\mathcal{V}_b := \log W_b = \sum_j a^j \mathcal{V}_b^{(j)} \quad (20)$$

and use similar power expansions for $\mathcal{V}_b^{\text{ren}}$ and $\mathcal{Z}_{bc} := Z_{bc} Z$. Then (2) implies

$$\begin{aligned} \mathcal{V}_1^{\text{ren}(1)} &= \mathcal{V}_1^{(1)} - \mathcal{Z}_{11}^{(1)} - \mathcal{Z}_{12}^{(1)} \\ \mathcal{V}_1^{\text{ren}(2)} &= \mathcal{V}_1^{(2)} + \mathcal{Z}_{12}^{(1)} (\mathcal{V}_1^{\text{ren}(1)} - \mathcal{V}_2^{\text{ren}(1)}) \\ &\quad - \mathcal{Z}_{11}^{(2)} - \mathcal{Z}_{12}^{(2)} + \mathcal{Z}_{11}^{(1)} \mathcal{Z}_{12}^{(1)} + \frac{1}{2} \left(\left(\mathcal{Z}_{11}^{(1)} \right)^2 + \left(\mathcal{Z}_{12}^{(1)} \right)^2 \right) . \end{aligned} \quad (21)$$

Therefore, $\mathcal{V}_1^{\text{ren}(2)}$ is given by the minimally subtracted first line of the r.h.s. of (21) and consequently $\log \mathcal{W}_1^{\text{ren}}$ by the minimally subtracted r.h.s. of (14), with (15) in mind, *plus*

$$a^2 \lim_{\epsilon \rightarrow 0} \mathcal{Z}_{12}^{(1)} (\mathcal{V}_1^{\text{ren}(1)} - \mathcal{V}_2^{\text{ren}(1)}) = -\frac{a^2}{2} \Gamma_{12}^{(1)} \frac{\partial}{\partial \epsilon} (\mathcal{V}_1^{\text{ren}(1)} - \mathcal{V}_2^{\text{ren}(1)})|_{\epsilon=0} . \quad (22)$$

Use has been made of $\mathcal{Z}_{12}^{(1)} = Z_{12}^{(1)} = -\frac{1}{2\epsilon} \Gamma_{12}^{(1)}$.⁶ Similar to the derivation of (19) we get

$$a^2 \lim_{\epsilon \rightarrow 0} \mathcal{Z}_{12}^{(1)} \left(\mathcal{V}_1^{\text{ren}(1)} - \mathcal{V}_2^{\text{ren}(1)} \right) = -\frac{a^2 \Gamma_{12}^{(1)}}{24} \left(L_{\hat{k}-1, \hat{l}-1}^3 + L_{\hat{k}+1, \hat{l}+1}^3 - L_{\hat{k}-1, \hat{k}+1}^3 - L_{\hat{l}-1, \hat{l}+1}^3 \right) + a^2 \cdot (\text{terms} \propto \log \mu^2) . \quad (23)$$

Now with (14), (15) and (23) we arrive at

$$\begin{aligned} \log \mathcal{W}_1^{\text{ren}} = a \cdot (\dots) &- \frac{a^2 \Gamma_{12}^{(1)}}{24} \left(L_{\hat{k}-1, \hat{l}-1}^3 + L_{\hat{k}+1, \hat{l}+1}^3 - L_{\hat{k}-1, \hat{k}+1}^3 - L_{\hat{l}-1, \hat{l}+1}^3 \right) \\ &- \frac{a^2 \Gamma_{\text{cusp}}^{(2)}}{8} \left(L_{\hat{k}+1, \hat{l}-1}^2 + L_{\hat{k}-1, \hat{l}+1}^2 - L_{\hat{k}-1, \hat{l}-1}^2 - L_{\hat{k}+1, \hat{l}+1}^2 + \sum_{k=1}^n L_{k-1, k+1}^2 \right) \\ &+ a^2 \cdot (\text{terms} \propto \log \mu^2) + a^2 \mathcal{R}_n^{(2)\text{ren}}(\mu^2, s) + \mathcal{O}(a^3) . \end{aligned} \quad (24)$$

Inserting this together with (19) into (9) and balance the order a^2 terms we get a RG equation for $\mathcal{R}_n^{(2)\text{ren}}$ (by this we denote the minimally subtracted part of $\mathcal{R}_n^{(2)}$ here).

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} \mathcal{R}_n^{(2)\text{ren}} = \frac{\Gamma_{\text{cusp}}^{(2)}}{2} &\left(L_{\hat{k}-1, \hat{k}+1} + L_{\hat{l}-1, \hat{l}+1} + L_{\hat{k}+1, \hat{l}-1} + L_{\hat{k}-1, \hat{l}+1} - L_{\hat{k}-1, \hat{l}-1} - L_{\hat{k}+1, \hat{l}+1} \right) \\ &+ \frac{\Gamma_{12}^{(1)}}{2} \left(L_{\hat{k}-1, \hat{l}-1}^2 + L_{\hat{k}+1, \hat{l}+1}^2 - L_{\hat{k}-1, \hat{k}+1}^2 - L_{\hat{l}-1, \hat{l}+1}^2 \right) - \Gamma_{11}^{(2)} + \dots , \end{aligned} \quad (25)$$

where the dots stand for terms independent of μ^2 . They include $-\Gamma_{12}^{(2)}$, which as $-\Gamma_{12}^{(1)}$ should be independent of μ^2 . The only interesting unknown entry on the r.h.s. is $\Gamma_{11}^{(2)}$. We expect the situation to be similar to the cusp anomalous dimension, where in the light-like limit one can factor off a linear dependence on $\log(-2\mu^2 p_{k-1} p_k)$. Assuming such a behaviour also for the crossing matrix entries, we get $\Gamma_{11}^{(2)} = \gamma_{11}^{(2)} (L_{\hat{k}-1, \hat{k}+1} + L_{\hat{l}-1, \hat{l}+1})$ with a number $\gamma_{11}^{(2)}$ that has to be determined in a two-loop calculation. Then integration of (25) yields

$$\begin{aligned} \mathcal{R}_n^{(2)\text{ren}} = \frac{\Gamma_{\text{cusp}}^{(2)}}{8} &\left(L_{\hat{k}-1, \hat{k}+1}^2 + L_{\hat{l}-1, \hat{l}+1}^2 + L_{\hat{k}+1, \hat{l}-1}^2 + L_{\hat{k}-1, \hat{l}+1}^2 - L_{\hat{k}-1, \hat{l}-1}^2 - L_{\hat{k}+1, \hat{l}+1}^2 \right) \\ &+ \frac{\Gamma_{12}^{(1)}}{12} \left(L_{\hat{k}-1, \hat{l}-1}^3 + L_{\hat{k}+1, \hat{l}+1}^3 - L_{\hat{k}-1, \hat{k}+1}^3 - L_{\hat{l}-1, \hat{l}+1}^3 \right) \\ &- \frac{\gamma_{11}^{(2)}}{4} \left(L_{\hat{k}-1, \hat{k}+1}^2 + L_{\hat{l}-1, \hat{l}+1}^2 \right) + \mathcal{O}(\log \mu^2) . \end{aligned} \quad (26)$$

This is the two loop remainder function renormalised to accommodate the extra divergences due to the self-crossing. Since μ^2 in the dimensionally regularised $\mathcal{R}_n^{(2)\text{ren}}$

⁶Note that the contribution from (22) has been omitted in [14]. However, taking it properly into account would modify that result at the end by a factor 2 only.

originates from the expansion of $g^4 \mu^{4\epsilon}$, one can backward reconstruct the unrenormalised remainder function

$$\begin{aligned}
\mathcal{R}_n^{(2)}(\mu^2, \epsilon, s) = & \frac{\Gamma_{\text{cusp}}^{(2)}}{16\epsilon^2} \left((-\mu^2 s_{\hat{k}-1, \hat{k}+1})^{2\epsilon} + (-\mu^2 s_{\hat{l}-1, \hat{l}+1})^{2\epsilon} + (-\mu^2 s_{\hat{k}+1, \hat{l}-1})^{2\epsilon} \right. \\
& \left. + (-\mu^2 s_{\hat{k}-1, \hat{l}+1})^{2\epsilon} - (-\mu^2 s_{\hat{k}-1, \hat{l}-1})^{2\epsilon} - (-\mu^2 s_{\hat{k}+1, \hat{l}+1})^{2\epsilon} \right) \\
& + \frac{\Gamma_{12}^{(1)}}{16\epsilon^3} \left((-\mu^2 s_{\hat{k}-1, \hat{l}-1})^{2\epsilon} + (-\mu^2 s_{\hat{k}+1, \hat{l}+1})^{2\epsilon} \right. \\
& \left. - (-\mu^2 s_{\hat{k}-1, \hat{k}+1})^{2\epsilon} - (-\mu^2 s_{\hat{l}-1, \hat{l}+1})^{2\epsilon} \right) \\
& - \frac{\gamma_{11}^{(2)}}{8\epsilon^2} \left((-\mu^2 s_{\hat{k}-1, \hat{k}+1})^{2\epsilon} + (-\mu^2 s_{\hat{l}-1, \hat{l}+1})^{2\epsilon} \right) + \mathcal{O}\left(\frac{1}{\epsilon}\right). \quad (27)
\end{aligned}$$

Expanding the exponents and inserting $\Gamma_{12}^{(1)}$ from (7),(8) we finally get

$$\mathcal{R}_n^{(2)} = \frac{1}{8\epsilon^2} \left(\log \frac{s_{\hat{k}-1, \hat{l}+1} s_{\hat{k}+1, \hat{l}-1}}{s_{\hat{k}-1, \hat{l}-1} s_{\hat{k}+1, \hat{l}+1}} \log \frac{s_{\hat{k}-1, \hat{l}-1} s_{\hat{k}+1, \hat{l}+1}}{s_{\hat{l}-1, \hat{l}+1} s_{\hat{k}+1, \hat{k}-1}} - 2\gamma_{11}^{(2)} + \Gamma_{\text{cusp}}^{(2)} \right) + \mathcal{O}\left(\frac{1}{\epsilon}\right). \quad (28)$$

If instead of dimensional regularisation one uses a point splitting regularisation $x_{\hat{k}} = x_{\hat{l}} + \delta \cdot v$ (v some unit vector) the leading divergences coincide, if one identifies $\frac{1}{\epsilon^2}$ with $\log^2(1/\delta^2)$.

Therefore, the two loop remainder function for a null n -gon, while being finite in generic configurations, develops a \log^2 divergence in the distance, if two vertices approach each other. The prefactor of this divergence depends on cross-ratios formed out of the four neighbour vertices and is given by $1/8$ times the expression in brackets in (28).

For notational shortness, above we have been sloppy with indicating all the arguments on which the remainder in different formulæ depends. We end this section by summarising the complete pattern:

$$\begin{aligned}
\mathcal{R}_n^{(2)}(\mu^2, \epsilon, s_{kl}) &= \mathcal{R}_n^{(2)}(u_{kl}) + \mathcal{O}(\epsilon) \\
\mathcal{R}_n^{(2)}(\mu^2, \epsilon, \{s_{kl}\}) &= \frac{1}{\epsilon^2} H_n(\{u_{kl}\}) + \frac{1}{\epsilon}(\dots) + \mathcal{R}_n^{(2)\text{ren}}(\mu^2, \{s_{kl}\}) + \mathcal{O}(\epsilon) \\
\mathcal{R}_n^{(2)}(u_{kl}) &= \log^2(\delta^2) G_n(\{u_{kl}\}) + \mathcal{O}(\log \delta^2). \quad (29)
\end{aligned}$$

$\{s_{kl}\}$ and $\{u_{kl}\}$ denote the set of Mandelstam variables and cross-ratios in the self-crossing limit. Finally we have used $H_n = G_n$.

4 Octagon

As an example, we now specialise to the octagon and chose $\hat{k} = 1$ and $\hat{l} = 5$. The configuration of an octagon (in every dimension) can be described using at most 12 conformally invariant cross ratios. In four dimension there are only 9 independent cross ratios due to Gram constraints. So far it has not been possible to disentangle

these relations for four dimensions. So we use the usual choice for the 12 conformal cross ratios

$$u_{ij} = \frac{(x_i - x_{j+1})^2 (x_{i+1} - x_j)^2}{(x_i - x_j)^2 (x_{i+1} - x_{j+1})^2} . \quad (30)$$

Let us look at these cross ratios in the limit $x_1 = x_5 + \delta v$, $\delta \rightarrow 0$, when the loop becomes self-intersecting. We want to express a divergence in δ as a divergence in terms of conformal invariants. This relation will also contain Mandelstam variables (which are not conformally invariant) because distances are also not conformally invariant.

In the aforementioned limit we encounter three classes of cross ratios. The ratios $u_{26}, u_{27}, u_{36}, u_{37}$ are not affected by this limit and remain untouched. Four cross ratios $u_{14}, u_{15}, u_{48}, u_{58}$ remain finite (in the general case) but depend on the direction v . For example one finds

$$u_{14} = \frac{v^2 (x_2 - x_4)^2}{4vp_4 \cdot vp_1} . \quad (31)$$

The last class diverges as we approach the crossing situation $u_{16}, u_{25}, u_{38}, u_{47}$, e.g.

$$u_{16} = -\frac{1}{\delta} \frac{(x_2 - x_6)^2 (x_1 - x_7)^2}{2vp_5 (x_2 - x_7)^2} . \quad (32)$$

We can eliminate the dependence on the direction of v by considering combinations of various u_{kl} and find the relation

$$4 \log \delta = -\log(u_{47} u_{38} u_{25} u_{16}) + \log \left(\frac{s_{48} s_{57} s_{13} s_{26} s_{35} s_{17}}{s_{47} s_{38} s_{36} s_{27} v^4} \right) - \log(u_{15} u_{48}) \quad (33)$$

for the crossing limit. The first term on the r.h.s. of (33) is conformally invariant and becomes divergent in the limit. The other two terms stay finite and balance the conformal non-invariance of the l.h.s.. Finally with the abbreviation $u := u_{47} u_{38} u_{25} u_{16}$ we get from (28) and the discussion at the end of the previous section

$$\mathcal{R}_8^{(2)} = \frac{1}{32} \log^2 u \left(\log \frac{s_{86} s_{24}}{s_{84} s_{26}} \log \frac{s_{48} s_{26}}{s_{46} s_{28}} - 2\gamma_{11}^{(2)} + \Gamma_{\text{cusp}}^{(2)} \right) + \mathcal{O}(\log u) \quad (34)$$

for $x_1 \rightarrow x_5$. This is valid as long as the vector v defining the direction of the approach is not light-like and has a nonzero scalar product with p_1, p_4, p_5, p_8 .

5 Conclusions

With RG techniques we have calculated the leading divergence of the two loop remainder function in the limit of two approaching vertices of the null polygon. We found a behaviour $\propto \frac{1}{2} \log^2 \delta$, where δ measures the vanishing distance between the approaching vertices. The prefactor of this divergence is given by the product of two logarithms of cross-ratios parametrising the conformal geometry of the self-crossing plus some pure number. Only the determination of this number requires two loop calculations, all other ingredients are fixed by the well-known one loop structure of the matrix of anomalous dimensions.

The prefactor itself becomes logarithmically divergent if the self-crossing configuration degenerates to the crossing of two smooth pieces of the Wilson loop contour. In the octagon case, for example, such a situation would arise if p_4 and p_5 as well as p_8 and p_1 become collinear (i.e. $s_{46}, s_{28} \rightarrow 0$). This reflects the \log^3 divergence found in [14] for self-intersections at interior points of the edges of the polygon.

Our result could be checked independently by direct study of the corresponding limit in the Feynman diagrams responsible for the extra divergences in the self-crossing configuration. But even when the full two loop remainder will be available, the RG technique again can be used to get information on such special limits one order higher.

Note added:

In the original version of this paper we had used another translation factor between dimensional and point splitting regularisation. This led in (34) to a factor $1/128$ instead of $1/32$. Strong arguments for the translation rule used now are given in our recent paper [22].

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